CHAPTER 1. SIGNALS AND LINEAR SYSTEMS

Figure 1.2: The sinc signal

Figure 1.3: Various Fourier series approximations for the rectangular pulse in Illustrative Problem 1.1
2. Note that $x_n$ is always real. Therefore, depending on its sign, the phase is either zero or $\pi$. The magnitude of the $x_n$’s is $\frac{1}{2} \left| \text{sinc} \left( \frac{n}{2} \right) \right|$. The discrete spectrum is shown in Figure 1.4.

![Figure 1.4: The discrete spectrum of the signal in Illustrative Problem 1.1](image)

The MATLAB script for plotting the discrete spectrum of the signal is given next.

```matlab
% MATLAB script for Illustrative Problem 1.1.
n=[-20:1:20];
x_actual=abs(sinc(n/2));
figure
stem(n,x_actual);
```

When the signal $x(t)$ is described on one period between $a$ and $b$, as shown in Figure 1.5, and the signal in the interval $[a, b]$ is given in an m-file, the Fourier series coefficients can be obtained using the m-file fseries.m given next.

```matlab
function xx=fseries(funfcn,a,b,n,tol,p1,p2,p3)
% FSERIES     Returns the Fourier series coefficients.
% XX=FSERIES(FUNFCN,A,B,N,TOL,P1,P2,P3)
% funfcn=the given function, in an m-file.
% It can depend on up to three parameters
% p1,p2, and p3. The function is given
% over one period extending from 'a' to 'b'
```

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Editorial review has deemed that any suppressed content does not materially affect the overall learning experience. Cengage Learning reserves the right to remove additional content at any time if subsequent rights restrictions require it.
Illustrative Problem 1.2 [The Magnitude and the Phase Spectra] Determine and plot the discrete magnitude and phase spectra of the periodic signal $x(t)$ with a period equal to 8 and defined as $x(t) = \Lambda(t)$ for $|t| \leq 4$.

\begin{center}
\textbf{SOLUTION}
\end{center}

Because the signal is given by an m-file lambda.m, we can choose the interval $[a, b] = [-4, 4]$ and determine the coefficients. Note that the m-file fseries.m determines the Fourier series coefficients for nonnegative values of $n$, but because here $x(t)$ is real valued, we have $x_{-n} = x_n^*$. In Figure 1.6 the magnitude and the phase spectra of this signal are plotted for a choice of $n = 24$.

The MATLAB script for determining and plotting the magnitude and the phase spectra is given next.
1.2. FOURIER SERIES

Figure 1.6: The magnitude and the phase spectra in Illustrative Problem 1.2

```matlab
% MATLAB script for Illustrative Problem 1.2.
echo on
fnct='lambda';
a=-4;
b=4;
n=24;
```
\[ x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \]

in the interval \([-6, 6]\). A plot of this signal is shown in Figure 1.7.

![Figure 1.7: The periodic signal in Illustrative Problem 1.3](image)

**ILLUSTRATIVE PROBLEM**

**Illustrative Problem 1.3** [The Magnitude and the Phase Spectra] Determine and plot the magnitude and the phase spectra of a periodic signal with a period equal to 12 that is given by

\[ x(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \]

in the interval \([-6, 6]\). A plot of this signal is shown in Figure 1.7.

**SOLUTION**

The signal is equal to the density function of a zero-mean unit-variance Gaussian (normal) random variable given in the m-file normal.m. This file requires two parameters, \( \mu \) and \( \sigma \), the mean and the standard deviation of the random variable, which in the
1.2. FOURIER SERIES

problem are 0 and 1, respectively. Therefore, we can use the following MATLAB script to obtain the magnitude and the phase plots shown in Figure 1.8.

Figure 1.8: The magnitude and phase spectra in Illustrative Problem 1.3
1.2.1 Periodic Signals and LTI Systems

When a periodic signal $x(t)$ is passed through an LTI system, as shown in Figure 1.9, the output signal $y(t)$ is also periodic, usually with the same period as the input signal (why?), and therefore it has a Fourier series expansion.

If $x(t)$ and $y(t)$ are expanded as

$$x(t) = \sum_{n=-\infty}^{\infty} x_n e^{j2\pi nt/T_0}$$

(1.2.26)

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0}$$

(1.2.27)

Figure 1.9: Periodic signals through LTI systems

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$^2$We say *usually* with the same period as the input signal. Can you give an example where the period of the output is different from the period of the input?
then the relation between the Fourier series coefficients of \( x(t) \) and \( y(t) \) can be obtained by employing the convolution integral

\[
y(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) \, d\tau
\]

\[
= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_n e^{j2\pi n (t-\tau)/T_0} h(\tau) \, d\tau
\]

\[
= \sum_{n=-\infty}^{\infty} x_n \left( \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi n \tau/T_0} \, d\tau \right) e^{j2\pi nt/T_0}
\]

\[
= \sum_{n=-\infty}^{\infty} y_n e^{j2\pi nt/T_0}
\]

From the preceding relation we have

\[
y_n = x_n H \left( \frac{n}{T_0} \right)
\]

where \( H(f) \) denotes the transfer function\(^3\) of the LTI system given as the Fourier transform of its impulse response \( h(t) \):

\[
H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} \, dt
\]

---

**ILLUSTRATIVE PROBLEM**

**Illustrative Problem 1.4 [Filtering of Periodic Signals]** A triangular pulse train \( x(t) \) with period \( T_0 = 2 \) is defined in one period as

\[
A(t) = \begin{cases} 
  t + 1, & -1 \leq t < 0 \\
  -t + 1, & 0 < t < 1 \\
  0, & \text{otherwise}
\end{cases}
\]

1. Determine the Fourier series coefficients of \( x(t) \).

2. Plot the discrete spectrum of \( x(t) \).

3. Assuming that this signal passes through an LTI system whose impulse response is given by

\[
h(t) = \begin{cases} 
  t, & 0 \leq t < 1 \\
  0, & \text{otherwise}
\end{cases}
\]

plot the discrete spectrum and the output \( y(t) \). Plots of \( x(t) \) and \( h(t) \) are given in Figure 1.10.

---

\(^3\)Also known as the frequency response of the system.
1. We have

\[ x_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j2\pi nt/T_0} \, dt \]  

(1.2.33)

\[ = \frac{1}{2} \int_{-1}^{1} \Lambda(t) e^{-j\pi nt} \, dt \]  

(1.2.34)

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \Lambda(t) e^{-j\pi nt} \, dt \]  

(1.2.35)

\[ = \frac{1}{2} \mathcal{F}[\Lambda(t)]_{f=n/2} \]  

(1.2.36)

\[ = \frac{1}{2} \text{sinc}^2 \left( \frac{n}{2} \right) \]  

(1.2.37)

where we have used the facts that \( \Lambda(t) \) vanishes outside the \([-1, 1]\) interval and that the Fourier transform of \( \Lambda(t) \) is \( \text{sinc}^2(f) \). This result can also be obtained by using the expression for \( \Lambda(t) \) and integrating by parts. Obviously, we have \( x_n = 0 \) for all even values of \( n \) except for \( n = 0 \).

2. A plot of the discrete spectrum of \( x(t) \) is shown in Figure 1.11.

3. First we have to derive \( H(f) \), the transfer function of the system. Although this can be done analytically, we will adopt a numerical approach. The resulting magnitude of the transfer function and also the magnitude of \( H(n/T_0) = H(n/2) \) are shown in Figure 1.12. To derive the discrete spectrum of the output we employ the relation

\[ y_n = x_n H \left( \frac{n}{T_0} \right) \]  

(1.2.38)

\[ = \frac{1}{2} \text{sinc}^2 \left( \frac{n}{2} \right) H \left( \frac{n}{2} \right) \]  

(1.2.39)

The resulting discrete spectrum of the output is shown in Figure 1.13.
1.2. **FOURIER SERIES**

![Graph showing discrete spectrum of a signal](image)

Figure 1.11: The discrete spectrum of the signal

The MATLAB script for this problem follows.

```
%% MATLAB script for Illustrative Problem 1.4.
echo on
n=[-20:1:20];
%% Fourier series coefficients of x(t) vector
x=.5*(sin(n/2)).^2;
%% sampling interval
ts=1/40;
%% time vector
t=[-.5:ts:.5];
%% impulse response
fs=1/hs;
h=[zeros(1,20),t(21:61),zeros(1,20)];
%% transfer function
H=fft(h)/fs;
%% frequency resolution
df=fs/80;
f=[0:df:fs]-fs/2;
%% rearrange H
H1=fftshift(H);
y=x.*H1(21:61);
%% Plotting commands follow.
```

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1.3 Fourier Transforms

The Fourier transform is the extension of the Fourier series to nonperiodic signals. The Fourier transform of a signal $x(t)$ that satisfies certain conditions, known as Dirichlet’s